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AMBIGUOUS BEHAVIOR OF LOGIC BISTABLE SYSTEMS**

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ABSTRACT

The standard specifications of logic bistable devices do not specify the behavior under conditions in which the input is logically undefined or in which certain kinds of multiple input changes occur. These conditions are unavoidable in logic synchronizers and arbiters. A general deterministic model of bistable devices is proposed, consisting of a nonlinear differential system with some adequate properties. Analysis of this model shows that bistable devices can be driven into a logically undefined region by certain admissible inputs and can remain in this region for an unbounded length of time.

1. INTRODUCTION

This paper is concerned with the behavior of logic bistable elements under input conditions that occur unavoidably in synchronizers that mediate communications between systems that do not share a common time reference, or in asynchronous arbiters that must allocate a resource to one of several users whose competing requests may occur at the same time. The input may be a weak or short pulse, or certain kinds of multiple input changes may occur more or less simultaneously. The resulting misbehavior of these circuits [1] produces serious system control errors. The paper is based on part of the doctoral dissertation written by the first author [2], in which the following concepts are applied to tunnel-diode and transistor flip-flops. Here we consider only a noise-free or deterministic model of a bistable device; in [2] the effect of additive white noise is discussed qualitatively.

In Figures 1 and 2 we have reproduced bistable output waveforms reported in [1]. The pictures at the left have been obtained using a sampling scope (1000dots/division), while the traces at the right are real-time sample responses. The output hangs up at a logically undefined voltage level (MC1016) or oscillates within the logically undefined region under the given input conditions. The time until logical resolution can be much larger than the switching time with ideal inputs.

This behavior is particularly serious when the output of the bistable device must drive, in parallel, subsequent logical circuits; some circuits may be activated but not others, creating a conflict or paradox in the logical system.

Since there is experimental evidence that the larger resolution times are of low probability, the problem may be alleviated by providing a delay time, fixed or depending on a threshold sensing device.

2. MATHEMATICAL MODEL OF A BISTABLE SYSTEM

Our deterministic mathematical model for bistable elements consists of a system of time-invariant nonlinear differential equations driven by a

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piecewise continuous bounded function. It is assumed that the model exhibits the following fundamental properties:

(a) There exists a constant input function for which the system has two asymptotically stable equilibrium points (see, for example, [3], p. 58).

(b) The set of input functions contains a subset whose elements are able to steer the system, in a finite time interval, from a neighborhood of one of the equilibrium points to a neighborhood of the other, and a second subset whose elements are able to steer the system, in the same time interval, in the reverse direction. Those neighborhoods correspond to the "0" and "1" binary logical conditions of the bistable system. They will be referred to as *logical sets*.

In what follows and in the next sections we will be referring often to the concepts and the theorems of dynamical systems theory [3].

In state representation the model is expressed as the differential system

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t)] \qquad (1)$$

$$\underline{x}(0) = \underline{\xi}$$

where

(1) $\underline{x}(t) \in R^n$ represents the state of the system at time $t \in [0,T]$.

(2) $\underline{u} = \{\underline{u}(t),\ t \in [0,T]\}$ is an element in the set U of inputs consisting of *m*-vector-valued piecewise-continuous functions defined on the interval $[0,T]$ such that

$$a_i \leq u_i(t) \leq b_i, \quad i=1,2,\ldots,m$$

for any $t \in [0,T]$ where the a_i's and b_i's are real numbers. The Cartesian product of the sets $[a_i,b_i]$, $i=1,2,\ldots,m$ will be denoted by Ω.

(3) $\underline{f}: R^n \times \Omega \to R^n$ is a continuous mapping satisfying the conditions needed to guarantee that the solution $\psi[t,\underline{\xi},\underline{u}]$ of the system (1) is unique on $[0,T]$ for each $\underline{\xi} \in R^n$ and each $\underline{u} \in U$, and continuous on $[0,T]\times R^n \times U$ in the following sense:

(i) For $\underline{u}$ U, $t,s \in [0,T]$ and $\underline{\xi},\underline{\eta} \in R^n$, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |\psi_i[t,\underline{\xi},\underline{u}] - \psi_i[s,\underline{\eta},\underline{u}]| < \varepsilon \text{ whenever } |t-s| + \sum_{i=1}^{n} |\xi_i - \eta_i| < \delta$$

where ξ_i and η_i, $i=1,2,\ldots,n$, are the components of $\underline{\xi}$ and $\underline{\eta}$ respectively.

(ii) For $t \in [0,T]$, $\underline{\xi} \in R^n$ and $\underline{u},\underline{v} \in U$, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |\psi_i[t,\underline{\xi},\underline{u}] - \psi_i[t,\underline{\xi},\underline{v}]| < \varepsilon,$$

whenever

$$\sum_{i=1}^{m} \int_0^t |u_i(t) - v_i(t)|\,dt < \delta .$$

Furthermore, the following assumptions are included in order to satisfy the conditions for bistability.

(4) There exists a constant input function $\{\underline{c}, t \in [0,T]\} \in U$ that, extended in the obvious way to the entire real line R, determines the

autonomous differential equation

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t),\underline{c}] \tag{2}$$
$$\underline{x}(0) = \underline{\xi}$$

with unique and continuous solutions $\psi_0(t,\underline{\xi})$ on $R\times R^n$ and with two different asymptotically stable equilibrium points in R^n, namely, $\underline{e}_1$ and $\underline{e}_2$. The corresponding regions of attraction ([3],p. 57) are denoted respectively by A_1 and A_2. The system (2) is termed the *autonomous subsystem*.

(5) There exist subsets U_1 and U_2 in U, and neighborhoods $N(\underline{e}_1)\subset A_1$ and $N(\underline{e}_2)\subset A_2$ of $\underline{e}_1$ and $\underline{e}_2$ respectively, such that

$$K[T;\underline{\xi},U_1]\subset A_2$$

and

$$K[T;\underline{\eta},U_2]\subset A_1$$

for any $\underline{\xi}\in N(\underline{e}_1)$ and $\underline{\eta}\in N(\underline{e}_2)$. The set $K[T;\underline{\xi},U_1]$, defined as

$$K[T;\underline{\xi},U_1] = \bigcup_{\underline{u}\in U_1} \{\underline{x}\in R^n:\psi[t,\underline{\xi},\underline{u}] = x\}$$

is called the set of reachable states at T from $\underline{\xi}$ at 0, with respect to U_1. A similar definition corresponds to $K[T;\underline{\eta},U_2]$. The neighborhoods $N(\underline{e}_1)$ and $N(\underline{e}_2)$ are the logical sets.

3. EXISTENCE OF THE REGION OF INDECISEVENESS OF THE AUTONOMOUS SUBSYSTEM

If we define a map $\pi:R^n\times R\to R^n$ such that $\pi(\underline{x},t)=\psi_0(t,\underline{x})$, where $\psi_0(t,\underline{x})$ is the solution of the autonomous subsystem (2), it is clear that (2) is a dynamical system on R^n([3],p. 5). Therefore, the regions of attraction A_1 and A_2 are open invariant neighborhoods of $\underline{e}_1$ and $\underline{e}_2$, respectively ([3], p. 60). Furthermore, the boundary sets ∂A_1 and ∂A_2 of A_1 and A_2, respectively, are also invariant ([3], p. 13). We will show that there exists in R^n a nonempty set initiating solutions that do not approach either $\underline{e}_1$ or $\underline{e}_2$. Let d be any metric compatible with the usual R^n topology.

Proposition 1. The boundary sets ∂A_1 and ∂A_2 of A_1 and A_2, respectively, are nonempty closed sets.

Proof: The regions of attraction, A_1 and A_2, are disjoint since, otherwise, there exists a $\underline{x}\in R^n$ such that $\underline{x}\in A_1$ and $\underline{x}\in A_2$ and consequently $d[\psi_0(t,\underline{x}),\underline{e}_1]\to 0$ and $d[\psi_0(t_1\underline{x}),\underline{e}_2]\to 0$ as $t\to\infty$. This implies that $\underline{e}_1=\underline{e}_2$, which contradicts the assumption that $\underline{e}_1$ and $\underline{e}_2$ are different. Therefore, $A_1\cap A_2=\phi$. Furthermore, $A_1\cup A_2\neq R^n$ since A_1 and A_2 are open sets. Therefore ∂A_1 and ∂A_2 are nonempty. They are also closed by definition of a boundary set.

Proposition 2. ∂A_1 and ∂A_2 are unstable sets.

Proof: The proof is presented for A_1. Since A_1 is an open neighborhood of $\underline{e}_1$ in R^n, then $A_1\cap\partial A_1=\phi$, and $d[\partial A_1,\underline{e}_1]>0$. Let $S(\partial A_1,\varepsilon)$ denote the ε-neighborhood of ∂A_1 defined as the set

$$\{\underline{x}\in R^n:\ d[\underline{x},\underline{y}] < \varepsilon,\ \underline{y}\in\partial A_1\}$$

where $\varepsilon\in R^+$. We now choose a $\varepsilon\in R$ such that $d[S(\partial A_1,\varepsilon),\underline{e}_1]>0$ and assume that ∂A_1 is stable ([3],p. 84). Then, for any $\underline{z}\in\partial A_1$ there exists a $\gamma(\underline{z},\varepsilon)\in R$, $\gamma(\underline{z},\varepsilon)>0$, such that the positive semitrajectories of the γ-neighborhood of $\underline{z}$ are in $S(\partial A_1,\varepsilon)$; that is

$$\{\underline{x}\in R^n:\ \psi_0(\underline{y},t) = \underline{x},\ \underline{y}\in S(\underline{z},\gamma),\ t\in R^+\}\subset S(\partial A_1,\varepsilon).$$

But, $\underline{z}$ being a point in ∂A_1, its γ-neighborhood $S(z,\varepsilon)$ must have at least a point in A_1, say the point $\underline{w}$. Consequently, $d[\psi_0(\underline{w},t),\underline{e}_1]\to 0$ as $t\to\infty$.

Therefore

$$\{\underline{x} \in R^n: \psi_0(\underline{w},t) = \underline{x},\ t \in R^+\} \not\subset S(\partial A_1, \varepsilon),$$

which is a contradiction. Therefore A_1 is unstable.

Proposition 3. Given $\underline{x} \in A_1$, $\underline{x} \neq \underline{e}_1$, then the negative limit set, $L^-(\underline{x})$ ([3], p. 19), is included in the boundary set ∂A_1.

Proof: By a theorem ([3], p. 90) of dynamical system theory, the first negative prolongation limit set of $\underline{x}$, $J^-(\underline{x})$ ([3], p. 24), and A_1 are disjoint since $\underline{x} \in A_1 - \underline{e}_1$. Therefore $L^-(\underline{x}) \cap A_1 = \phi$ since $L^-(\underline{x}) \subset J^-(\underline{x})$. Let $\underline{y} \in L^-(\underline{x})$; then there exists a sequence $\{t_n\} \in R$ with $t_n \to -\infty$, such that $\psi_0(\underline{x},t_n) \to \underline{y}$. The sequence $\{\psi_0(\underline{x},t_n)$: $n=1,2,\ldots\}$ is in A_1 since A_1 is an invariant set. Consequently $\underline{y}$ is in the closure $\bar{A}_1$ of A_1 and since $\underline{y}$ is not in A_1, then $\underline{y} \in \partial A_1$.

A similar proposition applies to the negative limit set of a point, other than $\underline{e}_2$, in the region of attraction A_2.

Proposition 4. Let $\partial A_1 \neq \partial A_2$. Then

(i) If $\partial A_1 \cap \partial A_2 \neq \phi$, $\partial A_1 \cap \partial A_2$ is a closed invariant set.

(ii) If $\partial A_1 \cap \partial A_2$ is a singleton, then it is an unstable equilibrium point.

(iii) If $\bar{A}_1$ and $\bar{A}_2$ are the closures of, respectively, A_1 and A_2, then $R^n - (\bar{A}_1 \cup \bar{A}_2)$ is also an invariant set.

Proof: (i) $\partial A_1 \cap \partial A_2$ is a closed invariant set because ∂A_1 and ∂A_2 are closed and invariant ([3], p. 12).

(ii) Let $\{\underline{x}\} = \partial A_1 \cap \partial A_2$; then $\psi_0(\underline{x},t) = \underline{x}$ for every $t \in R$ because of (i). Thus $\underline{x}$ is a critical point by definition. It is unstable since ∂A_1 and ∂A_2 are unstable because of Proposition 2.

(iii) $\bar{A}_1$, $\bar{A}_2$ and $\bar{A}_1 \cup \bar{A}_2$ are invariant ([3], p. 12). Therefore, $R^n - (\bar{A}_1 \cup \bar{A}_2)$ is also invariant ([3], p. 13).

Comments: (1) The propositions of this section can be extended in a straightforward manner to a case of autonomous dynamical system with more than two asymptotically stable equilibrium points.
(2) Since ∂A_1, ∂A_2 and $R^n - (\bar{A}_1 \cup \bar{A}_2)$ are invariant, the set $R^n - (A_1 \cup A_2)$ initiates trajectories of the autonomous subsystem not approaching either one of the asymptotically stable equilibrium points. This set is called the *region of indecisiveness*.
(3) If the region of indecisiveness is composed of only the boundary sets of the two regions of attraction, it is unstable.
(4) Trajectories of the autonomous subsystem starting in a region of attraction of one of the desired equilibrium points may require an arbitrarily long time to reach the logical set within such a region of attraction, since the boundary of a region of attraction is the negative limit set of the points in the region.

4. EXISTENCE OF ADMISSIBLE INPUTS DETERMINING AMBIGUOUS BEHAVIOR

Assuming that the bistable system state is initially at a point in the logical set $N(\underline{e}_1)$ $[N(\underline{e}_2)]$, and the input function applied in the interval $[0,T]$ belongs to the subset $U_1[U_2]$ and is followed by the constant input $\underline{c}$, the system state will change from the initial logical condition to the other, within the finite interval of duration T, and will permanently remain in this second condition. But if the input function is one that would bring the state of the system at T to a point in the region of indecisiveness, after T the state would remain in this region so that the system will not ever achieve a logical definition. In this section we show that there are indeed admissible input functions that will produce a permanent logically undefined condition of the bistable system.

We now consider the metric space composed by the input-function set, U, and the metric

$$d^1[\underline{u},\underline{v}] = \sum_{i=1}^{m} \int_0^T |u_i(t)-v_i(t)|dt \quad .$$

For the sake of simplicity of notation we will denote it also by U.

Proposition 5. The metric space U is connected.

Proof: Let $\underline{u}$ and $\underline{v}$ be any two elements in U. Define the mapping $\underline{h}$: $[0,1]\rightarrow U$ defined by

$$\underline{h}(s,t) = s\underline{u}(t) + (1-s)\underline{v}(t), \quad s\in[0,1]$$

Obviously $\underline{h}(0,t)=\underline{v}(t)$ and $\underline{h}(1,t)=\underline{u}(t)$, and for $s\in[0,1]$, $\underline{h}(s,\cdot)$ is in U. Furthermore, for given s and r in (0,1), we have

$$d^1[\underline{h}(x,\cdot),\underline{h}(r,\cdot)] = \sum_{i=1}^{m} \int_0^T |h_i(s,t)-h_i(r,t)|dt=|s-r| \; d^1[\underline{u},\underline{v}]$$

so that for a given $\varepsilon > 0$, there is a $\delta(\varepsilon) = \varepsilon/(d^1[\underline{u},\underline{v}])$ such that $|s-r| < \delta(\varepsilon)$ implies $d^1[\underline{h}(s,\cdot), h(r,\cdot)] < \varepsilon$. Also $\underline{h}(s,\cdot) \rightarrow \underline{v}$ as $s \rightarrow 0$ and $\underline{h}(s,\cdot) \rightarrow \underline{u}$ as $s \rightarrow 1$. Consequently the mapping $\underline{h}$ is continuous on [0,1], and the metric space U is arcwise connected and therefore connected.

Given an $\underline{\xi}\in R^n$ and a $T\in R$ we now define a mapping $\psi_{T,\underline{\xi}}$: $U\rightarrow R^n$ such that for a $\underline{u}$ in U $\psi_{T,\underline{\xi}}[\underline{u}] = \psi[T,\underline{\xi},\underline{u}]$, the solution of the bistable system (1). The image $\psi_{T,\underline{\xi}}[U]$ of the metric space U with respect to the mapping $\psi_{T,\underline{\xi}}$ is connected and further, arcwise connected, since the mapping is continuous. The connectivity of $\psi_{T,\underline{\xi}}(U)$ is used in the next proposition. Notice that $\psi_{T,\underline{\xi}}(U)$ is the set of reachable states at T from $\underline{\xi}$ at 0, with respect to U, that is, $\psi_{T,\underline{\xi}}(U) = K[T;\underline{\xi},U]$.

Proposition 6. If the state of the bistable system is in either $N(\underline{e}_1)$ $N(\underline{e}_2)$ at time 0, then there exist input functions in U that bring the state to the boundary sets of the regions of attraction A_1 and A_2 at time T.

Proof: We are going to prove the proposition for the set $N(\underline{e}_1)$. Let the state at time 0 be $\underline{\xi}\in N(\underline{e}_1)$. If the constant input function $\underline{c}$ in U is applied, the state of the system will remain in A_1 during all the time of application of such function, since $N(\underline{e}_1)$ in in A_1 and this set is invariant whenever $\underline{c}$ is applied. Thus $\psi_{T,\underline{\xi}}[\underline{c}]\subset A_1$. On the other hand, for $\underline{u}$ in U_1, $\psi_{T,\underline{\xi}}[\underline{u}]$ is in A_2, because of assumption (5) in Section 2. Therefore, we have in U elements $\underline{c}$ and $\underline{u}$ whose images due to $\psi_{T,\underline{\xi}}$ are in A_1 and A_2, respectively. Therefore, $\psi_{T,\underline{\xi}}[U]\cap A_1\neq\phi$ and $\psi_{T,\underline{\xi}}[U]\cap R^n-A_1\neq\phi$. Then, $\psi_{T,\underline{\xi}}[U]\cap \partial A_1\neq\phi$ ([4], Thm. 3.19.9). That is, there are some $\underline{u}\in U$ such that $\psi_{T,\underline{\xi}}[\underline{u}]\in \partial A_1$. Similarly, $\psi_{T,\underline{\xi}}[U]\cap A_2\neq\phi$ and $\psi_{T,\underline{\xi}}[U]\cap R^n-A_2\neq\phi$; therefore, $\psi_{T,\underline{\xi}}[U]\cap \partial A_2\neq\phi$. That is, there are some $\underline{v}\in U$ such that $\psi_{T,\underline{\xi}}[\underline{v}]\in \partial A_2$.

Proposition 7. Let the state of the bistable system at time 0 be in either $N(\underline{e}_1)$ or $N(\underline{e}_2)$. Assume that the closures $\bar{A}_1$ and $\bar{A}_2$ of the regions of attraction are disjoint; then there are input functions in U that bring the state to the subset $R^n-(\bar{A}_1\cup\bar{A}_2)$ of R^n at time T.

Proof: We are going to prove the proposition for $N(\underline{e}_1)$. Let $\underline{\xi}\in N(\underline{e}_1)$ be the system state at time 0, and E represent the set $\psi_{T,\underline{\xi}}[U]$. From the proof of the last proposition, we have $E\cap\bar{A}_1\neq\phi$ and $E\cap\bar{A}_2=\phi$. Now suppose that $E \cap (R^n - (\bar{A}_1 \cup \bar{A}_2)) = \phi$ so that $E = (E \cap \bar{A}_1) \cup (E \cap \bar{A}_2)$. But $E \cap \bar{A}_1$ and $E \cap \bar{A}_2$ are closed in E and disjoint. Consequently, E is not a connected set, which is a contradiction of what was found before. Therefore E or $\psi_{T,\underline{\xi}}[U]$ intersects $R^n - (\bar{A}_1 \cup \bar{A}_2)$ in a nonempty set. In turn this implies that there are some $\underline{u} \in U$ such that $\psi_{T,\underline{\xi}}[\underline{u}]$ is in $R^n - (\bar{A}_1 \cup \bar{A}_2)$.

5. COMMENTS AND CONCLUSIONS

In electronic circuits, in general, there is always present inherent noise. The reference [2] includes a discussion on the noise effect on the operation of bistable devices. Due to the noise, and because of the instability of either the region of indecisiveness or subsets of it, the state of bistable devices, when driven by certain admissible input functions, stay in a logically undefined condition for an unbounded length of time rather than forever.

Based on the results of the paper, we conclude that it is not possible to expect with certainty the achievement of a logical condition by bistable devices within a fixed time interval, when the allowed input signals are bounded but otherwise unrestricted.

6. ACKNOWLEDGEMENT

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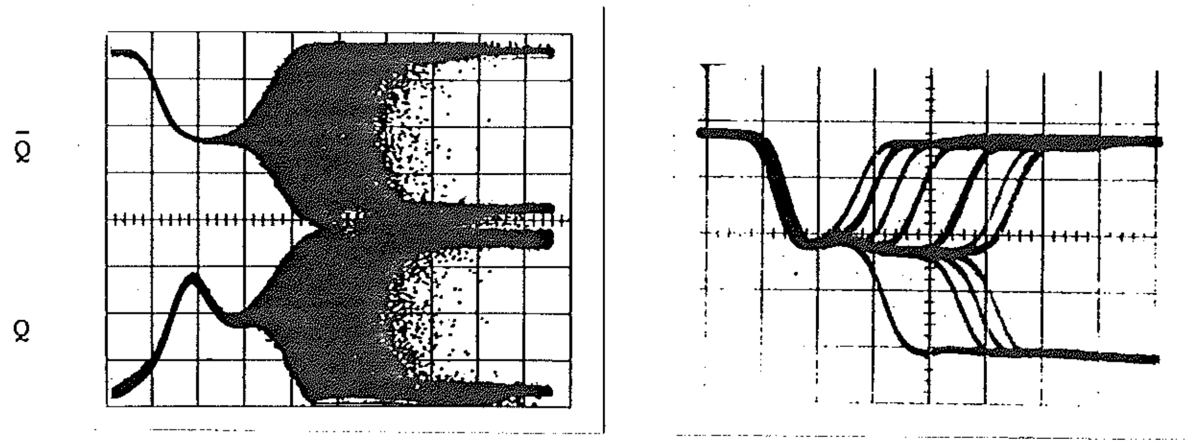

FIGURE 1. Sampling and real-time oscilloscope displays of the response of an ECL clocked R-S flip-flop (Motorola MC1016) to the clock input signal being switched off as the data input signal is changing. 5 nsec./div., 0.25V/div. for sampling displays; 10 nsec./div., 0.2V/div. for real-time displays.

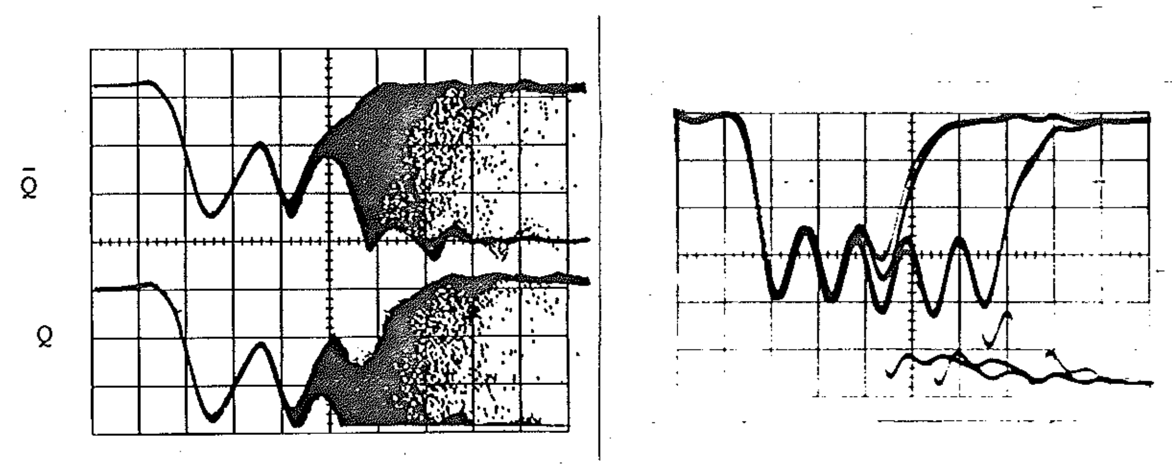

FIGURE 2. Sampling and real-time oscilloscope displays of the response of a TTL R-S flip-flop (constructed by cross-tying two SN7400 NAND gates) to both inputs being changed HIGH simultaneously (sampling displays) and to a weak pulse applied to one input (real-time displays). 5 nsec./div., 1V/div. for sampling displays; 10 nsec./div., 1V/div. for real-time displays.